647510 **MSC2MatC201x** **Seat No :________**

M.Sc.(Maths.) Semester - 2 (*CBCS*) Examination
March/April - 2025 (NEW COURSE)
Algebra 2 (Core)

Time: 2:30 Hours **Marks: 70**

Instructions:
1. All questions are compulsory.
2. Figures to the right indicate marks.

Que1(A) Answer any TWO of the following questions. (10)

(1) Show that a polynomial of degree n over a field can have at most n roots in any extension field.

(2) State and prove Eisenstein Criterion.

(3) Let $f(x) \in F[x]$. Then $f(x)$ has multiple roots if and only if $f(x)$ and $f'(x)$ have nontrivial common factor.

Que-1(B) Answer any ONE of the following questions. (04)

(1) Find the splitting field of the polynomial $p(x) = (x-1)^2(x^3-2) \in Q[x]$.

(2) Let K be an extension of F. If $a, b \in K$ are algebraic over F, then show that $a + b$ is also algebraic over F.

Oue-2(A) Answer any TWO of the following questions. (10)

(1) Let $p(x)$ be an irreducible polynomial in $F[x]$. Then there exists an extension E of F in which $p(x)$ has a root.

(2) If K is a finite extension of F, then $G(K,F)$ is a finite group with $o\big(G(K,f)\big) \leq [K:f]$.

(3) Let K be a normal extension of F, H be a subgroup of $G(K,F)$ and K_H be a fixed field of H. Then sow that $[K:K_H] = o(H)$.

Que-2(B) Answer any ONE of the following questions. (04)

(1) Compute $G(\mathbb{C}:\mathbb{R})$.

(2) State and prove Fundamental Theorem of Galios Theory.

Que-3(A) Answer any TWO of the following questions. (10)

(1) Let F be a field. Then show that there exists an extension $\bar{F}$ that is algebraic over F and is algebraically closed; that is, each field has an algebraic closure.

(2) Define Module and give two examples of it.

(3) If M is an R-module and $x \in M$, then show that the set $K = \{rx + nx | r \in R, n \in Z\}$ is an R-submodule of M containing x.

Que-3(B) Answer any ONE of the following questions. (04)

(1) Define Submodule. Let $(N_i)_{i\in\Lambda}$ be a family of R-submodules of an R-module M. Then show that $\bigcap_{i\in\Lambda} N_i$ is also an R-module.

(2) Define Submodule of a module of a module with an example. Show that the polynomial ring $R[x]$ over a ring R is an R −module.

Que-4(A) Answer any TWO of the following questions. (10)

(1) State and prove Fundamental Theorem of R-homomorphism.

(2) Let R be a ring with unity and M be an R module. Then Show that M is simple if and only if $M \neq \{0\}$ and M is generated by some $0 \neq x \in M$.

(3) The submodules of a quotient module M/N are of the form U/N, where U is a submodule of M containing N.

Que-4(B) Answer any ONE of the following questions. (04)

(1) State and prove Schur's Lemma.

(2) Let M be a finitely generated free module over a commutative ring R. Then show that all the bases of M have the same number of elements.

Que-5(A) Answer any TWO of the following questions. (10)

(1) Let $f: M \to N$ is an $R-$homomorphiism of an $R-$module M to an $R-$module N. Show that:

   i. $Ker\ (f)$ is an $R-$submodule of M.

   ii. $Im\ (f)$ is an $R-$submodule of N.

(2) Let $F \subset E$ and $f(x), g(x) \in F[x]$. If $f(x)$ and $g(x)$ have a nontrivial common factor in $E[x]$, then prove that they have a nontrivial common factor in $F[x]$.

(3) Let R be a ring with unity. Let $Hom_R(R,R)$ be the ring of endomorphisms of R regarded as a right R-module. Then show that $R \cong Hom_R(R,R)$ as rings.

Que-5(B) Answer any ONE of the following questions. (04)

(1) Let R be a ring with unity. An R-module M is cyclic if and only if $M \cong R/I$, for some left ideal I of R.

(2) Define the following terms:

   i. Finitely Generated Module

   ii. $R-$homomorphism of modules

   iii. Completely Reducible Module

   iv. Rank of a Module

****